PHYSICAL REVIEW E VOLUME 60, NUMBER 6 DECEMBER 1999

Controlling Hamiltonian chaos via Gaussian curvature

Atta Oloumi[1] and Denis Teychenné[2]
[1]National Biocomputation Center, Stanford University Medical Center, 701 Welch Road, Suite 1128, Palo Alto, California 94304
[2]Department of Engineering Physics and Mathematics, Helsinki University of Technology, P.O. Box 2200, FIN-02150 HUT, Finland

(Received 3 March 1999; revised manuscript received 14 July 1999)

We present a method allowing one to partly stabilize some chaotic Hamiltonians which have two degrees of freedom. The purpose of the method is to avoid the regions of $V(q_1,q_2)$ where its Gaussian curvature becomes negative. We show the stabilization of the Hénon-Heiles system, over a wide area, for the critical energy $E=\frac{1}{6}$. Total energy of the system varies only by a few percent. [S1063-651X(99)50512-6]

PACS number(s): 05.45.−a

To control chaos in nonlinear dynamical systems, Ott, Grebogi, and Yorke (OGY) [1] proposed to apply a small perturbation to stabilize an unstable periodic orbit. This method was developed for dissipative systems and has been succesfully applied in a variety of cases. Lai, Ding, and Grebogi (LDG) [2] extended the OGY method to Hamiltonian chaos. But in the Hamiltonian systems, the control of chaos is more difficult because there are no attractors and the search for chaotic behavior stretches to large areas of the phase space. In the scientific and engineering disciplines, dissipative systems are more commonly used than Hamiltonian systems. The latter occur in the physics of charged particules (such as plasma fusion, laser-plasma interaction, free electron laser, or particle accelerators), in astronomy (planetary motion), and in atomic physics (motion of atoms in molecules and crystals to describe molecular vibrations or molecular reactions, or motion of electrons in molecules) [3,4]. To avoid the appearance of chaotic behavior in a conservative system, the classical method consists of establishing a map of chaotic and regular regions in phase space and then choosing initial conditions in a regular area. However, this method presents a major disadvantage: for a certain nonlinear parameter value of the Hamiltonian, chaos becomes general and the regular areas disappear. The LDG method can be used in the chaotic region but only along one unstable periodic orbit, after an extremely long chaotic transient [2]. In another approach, Wu *et al.* [5] proposed to control Hamiltonian chaos of a periodically driven system with one degree of freedom, by an external field, but large intensities were required (from 40 to 60 % of the original driving force).

Our goal is to present a stabilization method for some chaotic Hamiltonians which have two degrees of freedom. With this aim, we consider the Gaussian curvature of the potential energy surface, $V(q_1,q_2)$, of the system as one source of chaos. In this article, we study the behavior of the Hamiltonian system following a change of the Hamiltonian to avoid the regions of $V(q_1,q_2)$ where its Gaussian curvature becomes negative. We call this avoidance of negative curvature regions of the potential energy (ANCRP). Two ways to do this are (i) to omit, from the Hamiltonian, the terms causing the negativity of the Gaussian curvature on regions of $V(q_1,q_2)$ where the curvature becomes negative, and (ii) to change the periodicity of periodic Hamiltonian so that it is restricted to regions of $V(q_1,q_2)$ with a positive curvature. The first method is used for the Hénon-Heiles system and the second for the sinusoidal Hamiltonian. Our aim is not only to avoid the negative curvature regions of $V(q_1,q_2)$, but more generally, to separate positive curvature regions from negative curvature regions. By this approach, we expect to stabilize the system (i.e., to obtain regular orbits) when the chaos is general and fills the quasitotality of the phase space. This method should operate in a large part of the phase space with an average energy variation of 10% or less.

The Gaussian curvature of $V(q_1,q_2)$, must not be confused with the curvature of trajectories of the phase space translated as geodesics on a Jacobi [6], Eisenhart [6,7] or Finsler [8] manifold. There is no connection between the curvature of $V(q_1,q_2)$, and the curvature of geodesics in the phase space. For example, in the case of the Hénon-Heiles Hamiltonian, the Gaussian curvature of $V(q_1,q_2)$, can be positive or negative, while the curvature of the Jacobi manifold is always positive [7] even when the system is chaotic. The sign of curvature on a Jacobi manifold is related to the sign of Laplacian of $V(q_1,q_2)$, and not to the sign of Gaussian curvature of $V(q_1,q_2)$. The chaotic behavior of Hamiltonian flows (viewed as geodesic flows in a manifold) can result from negative curvature on the Jacobi manifold [9] or from parametric resonance of geodesics due to curvature fluctuation [6]. The Finsler geometric indicator of chaos can discriminate between chaotic and regular orbits, i.e., between chaotic and regular regions of the phase space. For the Hénon-Heiles Hamiltonian, one-to-one correspondence has been demonstrated between Finsler geometry and chaos [10]. This method gives results already obtained with the usual tools (distribution of chaotic and regular regions in the phase space) and is not useful as a stabilization tool.

We study the Hénon-Heiles Hamiltonian because it is the paradigmatic model in the study of the Hamiltonian chaos. Moreover, one finds it in various applications (astronomy, accelerator physics, atomic physics, etc.) [4]. The Hénon-Heiles model is a Hamiltonian with two degrees of freedom and quadratic positive kinetic energy [11]. The system is conservative. Therefore,

$$H(q_1,q_2,p_1,p_2)=E, \tag{1}$$

with

$$E=T(p_1,p_2)+V(q_1,q_2), \tag{2}$$

where

$$T(p_1,p_2)=\frac{1}{2}(p_1^2+p_2^2) \tag{3}$$

and

$$V(q_1,q_2)=\frac{1}{2}(q_1^2+q_2^2)+q_1^2q_2-\frac{1}{3}q_2^3. \tag{4}$$

The nonlinear parameter E is the total energy of the system which drives chaos. When $E=\frac{1}{12}$, there are only a few chaotic orbits, for $E=\frac{1}{8}$ approximately half of the Poincaré section is filled with chaotic regions, and finally for $E=\frac{1}{6}$ the chaos is general and has invaded the quasitotality of phase space.

The Gaussian curvature K [12] of the potential of this Hamiltonian whose kinetic energy has a quadratic form, can be written as

$$K(q_1,q_2)=\frac{1-4q_1^2-4q_2^2}{[1+(q_1+2q_1q_2)^2+(q_1^2+q_2-q_2^2)^2]^2}, \tag{5}$$

which is positive inside a circle of equation $q_1^2+q_2^2=\frac{1}{4}$.

The Gaussian curvature K depends only on the space coordinates. In addition, negative curvature regions form compact blocks. The application of the stabilization method is therefore facilitated.

The Poincaré section shows chaotic areas for total energy $E=\frac{1}{8}$, with large regular islets [Fig. 1(a)]. In this situation, the classic stabilization method would be to choose the initial conditions in an islet or to ''push'' the orbit to an islet by an external energy contribution.

As the value of energy increases, the islets gradually disappear. Thus, for $E=\frac{1}{6}$ with the same initial conditions as those in Fig. 1(a), stable islets disappear and make way for a chaotic sea [Fig. 1(b)].

The ANCRP technique becomes a cutoff to the frontier between negative curvature areas and positive curvature ones,

$$H(q_1,q_2,p_1,p_2)=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(q_1^2+q_2^2)+q_1^2q_2-\frac{1}{3}q_2^3 \tag{6}$$

if $q_1^2+q_2^2<\frac{1}{4}$, and

$$H(q_1,q_2,p_1,p_2)=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(q_1^2+q_2^2) \tag{7}$$

if $q_1^2+q_2^2>\frac{1}{4}$.

This cutoff condition is very simple. It introduces a discontinuity and then a shock on the circle $q_1^2+q_2^2=\frac{1}{4}$. Figure 1(c) shows the Poincaré section for $E=\frac{1}{6}$ with the cutoff condition applied. Unlike Fig. 1(b), Fig. 1(c) demonstrates for the same initial conditions regular orbits, which is evidence of a quasidisappearance of chaos and a stabilization of phase space.

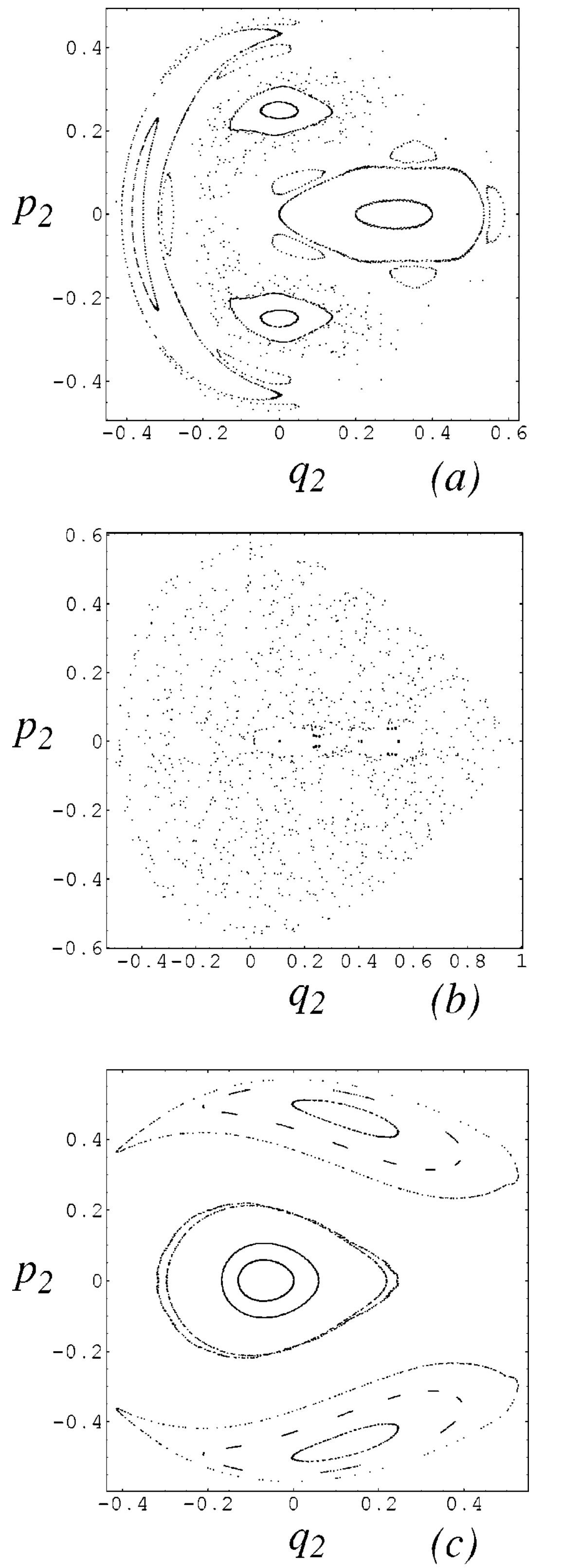

FIG. 1. Poincaré section of Hénon-Heiles Hamiltonian for (a) $E=\frac{1}{8}$ (regular), (b) $E=\frac{1}{6}$ (complete chaos), and (c) $E=\frac{1}{6}$, with cutoff stabilization method. Initial conditions: $q_1=0$ and $q_2=0$.

The condition of negative curvature region cutoff has not

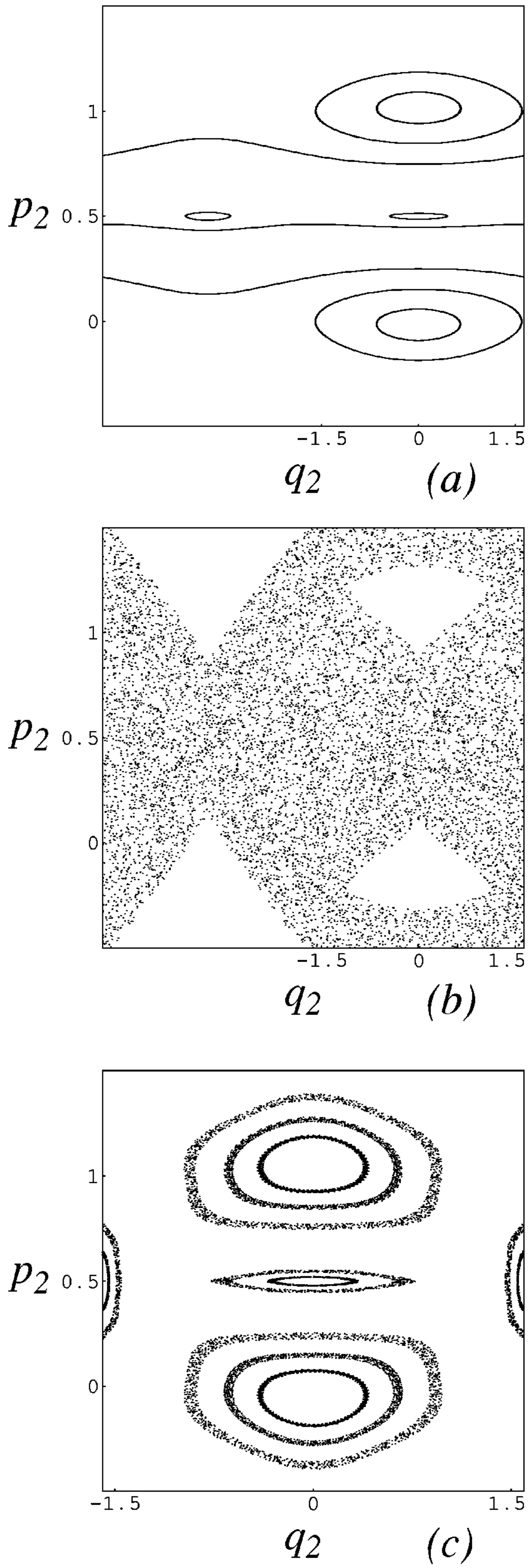

FIG. 2. Poincaré section of sinusoidal Hamiltonian for (a) $s=0.5$ (regular), (b) $s=1.5$ (complete chaos), and (c) $s=1.5$ with the stabilization method. Initial conditions are identical in the three cases.

rendered the Hamiltonian integrable. The attenuation or the disappearance of chaos results from a stabilization of orbits in the phase space.

What will happen if one moves the limit of the cutoff condition? Theoretically, by retreating the limit of the cutoff, a certain number of orbits have to become chaotic because they are submitted again to negative curvature regions. On the contrary, by decreasing the radius of the circle, conditions of shock are less violent because near the center the Hénon-Heiles potential is close to a quadratic potential. The regular regions must increase in size at the expense of the noise regions. By advancing or by retreating the limit of the cutoff, we observe, as planned, in one case the increase in the size of the three stable regions, and in another case the reappearance of chaos on the stable region edge.

As previously stated, the cutoff condition introduces a shock on the circle $q_1^2+q_2^2=\frac{1}{4}$. As has been shown by Sinai [13] in the case of the circular pool, repeated shocks on a circular edge are a source of chaos. So theoretically, the cutoff condition would have to introduce more chaos in the system. In fact it is the other way around and Poincaré section for the cutoff condition [Fig. 1(c)] shows the stabilization of the system with some appearance of noise. The cutoff condition has little effect on the total energy of the system, i.e., the average energy variation of the system is on the order of a few percent, depending on each orbit. The efficiency of this method, is not the result of an artificial diminution of the total energy.

We applied the ANCRP technique to other Hamiltonians, significant for their applications,

$$H=\frac{1}{2}p_1^2-\left(\frac{s}{4}\right)^2[\cos q_1+\cos(q_1-t)] \tag{8}$$

and the quartic oscillator Hamiltonian

$$H=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(q_1^4+q_2^4)+\frac{a}{2}q_1^2q_2^2. \tag{9}$$

The first Hamiltonian is time dependent but can be reduced to a 1.5 degree of freedom Hamiltonian system [14]. Specifically, let

$$V=-\left(\frac{s}{4}\right)^2[\cos q_1+\cos(q_1-q_2)], \tag{10}$$

where t becomes the second coordinate axis of the potential. This potential is a periodic function with a period 2π for q_2 and 2π for q_1. When the nonlinear parameter, $s>0.68$, large scale chaos appears in phase space. The Gaussian curvature of this potential

$$K=\frac{256a^4\cos q_1\cos(q_1-q_2)}{[256+a^4\sin^2 q_1+2a^4\sin q_1\sin(q_1-q_2)+2a^4\sin^2(q_1-q_2)]^2} \tag{11}$$

is periodic for q_1 and q_2 with a period 2π.

To avoid negative Gaussian curvature of this potential, which fills 50% of the potential surface, we impose a periodicity of π to the dynamics. To put this into practice, we choose one positive curvature region of the potential [including the point (0,0)], then we impose on the edges the following closing conditions:

(i) If $q_1>\frac{\pi}{2}$, $q_1=q_1-\pi$; $q_2=q_2-\pi$,

(ii) If $q_1<-\frac{\pi}{2}$, $q_1=q_1+\pi$; $q_2=q_2+\pi$,

(iii) If $q_2>q_1+\frac{\pi}{2}$, $q_2=q_2-\pi$,

(iv) If $q_2<q_1-\frac{\pi}{2}$, $q_2=q_2+\pi$.

These few conditions define a region of the potential with a positive Gaussian curvature. It is then possible to study the effect of the ANCRP technique. For $s=1.5$, phase-space is widely chaotic [Fig. 2(b)]. For the same initial conditions, the Hamiltonian with ANCRP technique, is stable [Fig. 2(c)]. We find again, as in the case of the Hénon-Heiles Hamiltonian, another success for the ANCRP technique.

The third Hamiltonian defined by Eq. (9), is almost completely chaotic for $a=12$. Gaussian curvature of the potential energy is

$$K=\frac{36q_1^2q_2^2-3a^2q_1^2q_2^2+6a(q_1^4+q_2^4)}{[1+(2q_1^3+aq_1q_2^2)^2+(2q_2^3+aq_2q_1^2)^2]^2}. \tag{12}$$

The negative regions broadly fill the potential surface with the shape of a four-leaved clover. It is not possible in this case to divide the surface of the potential between positive and negative regions, because it generates 16 shocks for a large number of trajectories at each period. Morever, the condition $a=0$ for the negative curvature regions (i.e., the use of a cutoff-like ANCRP technique, as with the Hénon-Heiles Hamiltonian) involves a very important variation of the energy of the system. Still, the quartic oscillator shows a strong correlation between chaos and Gaussian curvature of the potential: (i) the last stable trajectories are the ones which avoid the negative curvature regions, and (ii) large scale chaos appears in the system for $a>6$ [15] and we calculate [from Eq. (12)] the emergence of the negative curvature regions precisely for $a>6$. The same correlation can be found, for example, for another Hamiltonian of the quartic oscillator family,

$$H=\frac{1}{2}(p_1^2+p_2^2)+3q_1^4+q_2^4-aq_1^2q_2^2. \tag{13}$$

In this case, chaos and negative curvature of the potential appear both for $a>0$.

We can conclude that for the three different two-dimensional Hamiltonian systems, avoidance of negative curvature of the potential provides more stability. It is well known since Benettin *et al.* [16] that some trajectories crossing negative curvature regions of the potential remain stable. We show here that Gaussian curvature of the potential is strongly involved in the emergence of chaos, even if chaotic behavior and negative curvature of the potential are not equivalent.

The authors thank Jan Walleczek, Todd Pawlicki, and Jeffrey Carson from Stanford University for useful suggestions.

[1] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[2] Y. C. Lai, M. Ding, and C. Grebogi, Phys. Rev. E **47**, 86 (1993).
[3] A. J. Lichtenberg and M. A. Liberman, *Regular and Chaotic Dynamics*, Applied Mathematical Sciences (Springer-Verlag, New York, 1992).
[4] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Interdisciplinary Applied Mathematical (Springer-Verlag, New York, 1990).
[5] Z. Wu, Z. Zhu, and C. Zhang, Phys. Rev. E **57**, 366 (1998).
[6] M. Pettini and R. Valderatto, Chaos **5**, 646 (1995).
[7] M. Cerruti-Sola and M. Pettini, Phys. Rev. E **53**, 179 (1996).
[8] M. Di Bari and P. Cipriani, Planet. Space Sci. **46**, 1543 (1998).
[9] I. Arnold, *Les Méthodes Mathématiques de la Mécanique Classique* (MIR, Moscow, 1976).
[10] P. Cipriani and M. Di Bari, Phys. Rev. Lett. **81**, 5532 (1998).
[11] H. Hénon and C. Heiles, Astron. J. **69**, 73 (1964).
[12] I. N. Bronshtein and K. A. Semendyayev, *Handbook of Mathematics* (Springer, Berlin, 1997).
[13] Y. G. Sinai, Russ. Math. Surveys **25**, 137 (1970).
[14] D. F. Escande, Phys. Rep. **121**, 165 (1985).
[15] V. M. Bannur, P. K. Kaw, and J. C. Parikh, Phys. Rev. E **55**, 2525 (1997).
[16] G. Bennetin, R. Brambilla, and L. Galgani, Physica A **87**, 381 (1977).